Nonequivalence of phonon modes in the sine-Gordon equation

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We study the resonances in the sine-Gordon equation driven by an ac force using a linear perturbation theory. We show that resonances take place when the driving frequency δ is equal to half of the phonon modes' frequencies as has been shown numerically in our earlier work [N. R. Quintero, A. Sanchez, and F. G. Mertens, Phys. Rev. E **62**, R60 (2000)], however, we find that the ac force is able to excite not all the phonon modes, but rather only the odd phonons (i.e., the ones with odd eigenfunctions).

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I. INTRODUCTION

The $1+1$ dimensional sine-Gordon (sG) equation $[u_{xx}-u_{tt}=\sin u]$ appears in a wide variety of physical systems, including charge-density-wave materials, magnetic flux in Josephson lines, splay waves in membranes, Bloch wall motion in magnetic crystals, and models of elementary particles among others [2]. In most realistic physical contexts this completely integrable partial differential equation appears under the effect of damping and dc [3] or ac [1,4] driving.

In this work we focus on the resonances of the linearization spectrum of the sG equation under the effect of variable frequency ac driving in the presence of damping. In order to study the resonances of the sG system perturbed by an ac force, we investigate the approximate solution of the following equation by using a linear perturbation theory

$$\phi_{tt}-\phi_{xx}+\sin(\phi)=-\beta\phi_t+f(t), \tag{1}$$

where β and ϵ are small parameters, $f(t)=\epsilon\sin(\delta t+\delta_0)$ is an ac force with an amplitude ϵ, a frequency δ, and phase δ_0. Here we also include the effect of dissipation through the damping coefficient β. For this model it has been numerically observed in Ref. [1] that the energy of the system grows when the driving frequency δ is nearly half the frequency of the extended (phonon) eigenmodes pertaining to the continuous spectrum (due to resonance effects). In Ref. [1], following Ref. [5], the present authors used the dispersion relation $\omega_n\approx\sqrt{1+(2n\pi/L)^2}$, (where $n=1,2,\ldots,N-1$; $N=L/\Delta x$, Δx is the lattice spacing, and L is the length of the finite computational domain) for the frequencies of the phonon modes of Eq. (1); however, we will show that for the model of Eq. (1) considered with free boundary conditions, the dispersion relation should read $\omega_n\approx\sqrt{1+[(n-1)\pi/L]^2}$, $(n=1,2,\ldots)$ when $n/L\ll 1$. It was thus observed in Ref. [1] that when the sG system is driven by an ac force and the driving frequency is close to being half of the frequency of the odd n (in the last formula) phonon eigenfrequencies, these phonon modes are excited while, unexpectedly, the same was not true for the even eigenmodes. From this picture the main question that arises is why are the rest of the phonon modes (even n) not excited by the ac force?

The aim of this work is to analyze these resonance phenomena by using a linear perturbation theory. We will show why the ac driving is responsible for the ''preferrential'' excitation of only some (i.e., the ''odd'') of the phonon modes.

II. PERTURBATION THEORY

We assume that the solution of Eq. (1) has the form

$$\phi(x,t)=\phi_0(x-X(t))+\int_{-\infty}^{+\infty}dk A_k(t)f_k(x-X(t)), \tag{2}$$

where $\phi_0(x)$ is the exact static kink solution of the sG equation, and $f_k(x)$ are the eigenfunctions of the phonon modes with the corresponding frequencies ω_k,

$$f_k(x)=\frac{e^{ikx}}{\sqrt{2\pi}\omega_k}[k+i\tanh(x)],\qquad \omega_k=\sqrt{1+k^2}, \tag{3}$$

which along with the zero frequency $(\omega_b=0)$ Goldstone mode $f_b(x)=\partial\phi_0/\partial x=2/\cosh(x)$ form an orthonormal basis set (for more details see, e.g., Ref. [6]). The unknown, time dependent functions $X(t)$ and $A_k(t)\equiv[a_k(t)+ib_k(t)]/2$ represent the position of the center of the kink and the amplitude of phonon modes, respectively. If we impose free boundary conditions (FBC) on Eq. (1) at $\pm\infty$ and consider the static kink centered at $X(0)=0$, $\partial\phi(x,t)/\partial x$ in Eq. (2) should vanish at $\pm L/2\rightarrow\pm\infty$, $\dot{X}(0)=0$, $A_k(0)=0$, and $\dot{A}_k(0)=0$. Taking finite but large enough L we find that the FBC hold if

$$a_k\frac{\partial F_k}{\partial x}(\pm L/2)+b_k\frac{\partial G_k}{\partial x}(\pm L/2)=0, \tag{4}$$

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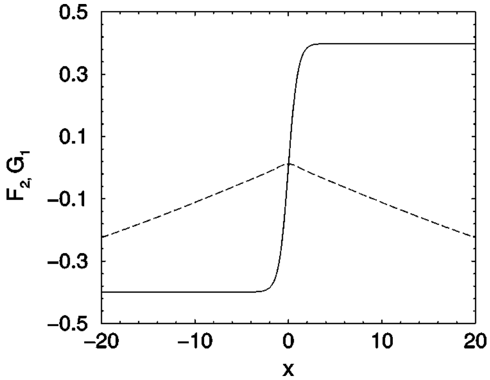

FIG. 1. We plot the spatial profile of the first odd (G_1) and even (F_2) eigenfunctions by solid and dashed lines, respectively.

where the functions $F_k(x)$ and $G_k(x)$ are the real and imaginary part of $f_k(x)$, respectively. Notice that, $F_k(x)$ is an even function, whereas $G_k(x)$ is an odd one (see in Fig. 1 the functions G_1 and F_2), and so $a_k(t)$ and $b_k(t)$ in Eq. (2) are related with the even and odd eigenfunctions of the phonon modes, respectively.

In order to obtain the equations for $X(t)$ and $A_k(t)$, we insert Eq. (2) into Eq. (1) and then project the obtained expression on the basis of eigenfunctions of the linearization operator $\{f_b, f_k\}$ using the orthogonality relations. After straightforward calculations we obtain the evolution equations for the kink's center,

$$\mathcal{L}_b X = -\frac{q f(t)}{M_0} + \frac{1}{M_0}\int dk[(\ddot{X}+\beta\dot{X})b_k + \dot{X}\dot{b}_k]N_1(k) - \frac{i}{4M_0}\int dk \int dk_1 (a_k b_{k_1} + a_{k_1} b_k) I_1(k,k_1), \tag{5}$$

where $\mathcal{L}_j \equiv \partial^2/\partial t^2 + \beta\partial/\partial t + \omega_j^2$ $(j=b,k)$ is the second order linear differential operator, $M_0=8$ is the kink's mass, and $q \equiv \int d\theta f_b = 2\pi$ is the topological charge of the sG kink (here and throughout the paper we omit the limits $\pm\infty$ in the integrals), and for the phonon amplitudes

$$\mathcal{L}_k A_k = f(t) N_2(k) - \dot{X}^2 i N_1(k) - i\int dk' \left[\frac{q f(t)}{M_0} A_{k'} - \dot{X}\dot{A}_{k'}\right] I_2(k',k) + \frac{i}{2}\int dk_1 \int dk_2 A_{k_1} A_{k_2}^{\star} I_3(k_1,k_2,k), \tag{6}$$

with

$$N_1(k) \equiv \frac{1}{i}\int d\theta f_b \frac{\partial f_k}{\partial \theta} = \sqrt{\frac{\pi}{2}} \frac{\omega_k}{\cosh\left(\frac{\pi k}{2}\right)},$$

$$N_2(k) \equiv \int d\theta f_k^{\star} = -\sqrt{\frac{\pi}{2}} \frac{\omega_k}{\sinh\left(\frac{\pi k}{2}\right)}, \quad k \neq 0,$$

$$I_1(k,k') \equiv \frac{1}{2i}\int d\theta \frac{\partial f_b^2}{\partial \theta} f_k f_{k'}^{\star} = \frac{(k^2 - k'^2)}{4\omega_k \omega_{k'} \sinh\left(\frac{\pi\Delta}{2}\right)},$$

$$I_2(k',k) \equiv \frac{1}{i}\int d\theta \frac{\partial f_k}{\partial \theta} f_{k'}^{\star}(\theta) = k\,\delta(k'-k) + I_1(k,k'),$$

$$I_3(k_1,k_2,k) \equiv \frac{1}{i}\int d\theta \frac{\partial f_b}{\partial \theta} f_{k_1}(\theta) f_{k_2}^{\star}(\theta) f_k^{\star}(\theta),$$

where $\Delta = k - k'$. Notice that in Eqs. (5) and (6) we have neglected the terms proportional to $\dot{X}^2 a_k$ and $\dot{X}^2 b_k$.

To solve the coupled nonlinear evolution equations (5) and (6) with the FBC (4), we use the fact that $\dot{X}(t)$ and $A_k(t)$ vanish when $\epsilon=0$ and so, if ϵ is a small parameter, we expand these two functions in powers of ϵ, i.e., $\dot{X}(t) = \Sigma_{j=1}^{\infty}\epsilon^j \dot{X}_j(t)$ and $A_k(t) = \Sigma_{j=1}^{\infty}\epsilon^j A_k^{(j)}(t)$ with the initial conditions $\dot{X}_j(0)=0$ and $\dot{A}_k^{(j)}(0)=0$. Inserting these two series into Eqs. (5), (6), and (4) we obtain a hierarchy of linear equations for the coefficients of these series. For the order ϵ and ϵ^2 these equations read

$$\ddot{X}_1 + \beta\dot{X}_1 = -\frac{q\sin(\delta t + \delta_0)}{M_0},$$

$$\ddot{a}_k^{(1)} + \beta\dot{a}_k^{(1)} + \omega_k^2 a_k^{(1)} = -\sqrt{\frac{\pi}{2}} \frac{\sin(\delta t + \delta_0)}{\omega_k \sinh\left(\frac{k\pi}{2}\right)},$$

$$\ddot{b}_k^{(1)} + \beta\dot{b}_k^{(1)} + \omega_k^2 b_k^{(1)} = \sin(\delta t + \delta_0)\int d\theta G_k = 0 \tag{7}$$

and

$$\ddot{X}_2 + \beta\dot{X}_2 = 0,$$

$$\ddot{a}_k^{(2)} + \beta\dot{a}_k^{(2)} + \omega_k^2 a_k^{(2)} = 0,$$

$$\ddot{b}_k^{(2)} + \beta\dot{b}_k^{(2)} + \omega_k^2 b_k^{(2)} = -\dot{X}_1^2 \sqrt{\frac{\pi}{2}} \frac{\omega_k}{\cosh\left(\frac{k\pi}{2}\right)} - \frac{q\sin(\delta t + \delta_0)}{M_0}\int dk'\, a_{k'}^{(1)} I_2(k',k) + \dot{X}_1 \int dk'\, \dot{a}_{k'}^{(1)} I_2(k',k) + \frac{1}{2}\int dk_1 \int dk_2\, a_{k_1}^{(1)} a_{k_2}^{(1)} I_3(k_1,k_2,k), \tag{8}$$

where $\{a_k^{(1)}, b_k^{(1)}\}$ and $\{a_k^{(2)}, b_k^{(2)}\}$ also satisfy Eq. (4). Since our purpose is to explain from an analytical point of view the resonances observed numerically in Ref. [1] in the undamped case, we continue the analysis of Eqs. (7) and (8) setting $\beta=0$. In this case the solution of Eq. (7) reads

$$\dot{X}_1=-\frac{q}{M_0\delta}[\cos(\delta t+\delta_0)-\cos(\delta_0)],$$

$$a_k^{(1)}=c_1\sin(\omega_k t)+c_2\cos(\omega_k t)+r_k\sin(\delta t+\delta_0),$$

$$c_1=-\frac{\delta r_k\cos(\delta_0)}{\omega_k},\quad c_2=-r_k\sin(\delta_0),$$

$$r_k=\frac{\sqrt{2\pi}}{2\omega_k(\delta^2-\omega_k^2)\sinh(k\pi/2)},\tag{9}$$

and $b_k^{(1)}$ is zero. Notice that when δ is close to ω_k the expansion for A_k is not valid since $a_k^{(1)}$ goes to infinity, so when $\delta\approx\omega_k$, the ac force excites the even modes, giving rise to the resonances in the system. Also, notice that, at least in the first order correction, there are no resonances at $\delta\approx\omega_k/2$.

It is interesting to note that not all phonon modes are solutions of the starting problem (1) with FBC (4). Indeed, by inserting this solution in Eq. (4) we obtain

$$a_k^{(1)}[\sin(kL/2)[k^2+\cosh^{-2}(L/2)]+k\cos(kL/2)\tanh(L/2)]=0.\tag{10}$$

This equation yields the allowed values of the wave number $k=k_m$ $(m=1,2,\ldots,N)$, so in the above equations we should change the integral over k to a sum over the N wave numbers. If we consider L large enough in Eq. (10), we find that if

$$k_m=\frac{(2m-1)\pi}{L},\tag{11}$$

Eq. (10) is approximately equal to zero for the first modes, however, for the largest values of $k=k_m$, $a_k^{(1)}$ should vanish [7]. Coming back to the evolution equations (8), and substituting the solutions (9) in the right-hand side of (8), we obtain that $\dot{X}_2=0$, $a_k^{(2)}=0$, and the equation for $b_k^{(2)}$ reads

$$\begin{aligned}\ddot{b}_k^{(2)}+\omega_k^2 b_k^{(2)}=&-\frac{\sqrt{2\pi}\omega_k}{4\cosh\left(\frac{k\pi}{2}\right)}\frac{q^2}{M_0^2\delta^2}[2+\cos(2\delta_0)\\&+\cos(2\delta t+2\delta_0)-4\cos(\delta_0)\cos(\delta t+\delta_0)]\\&-\frac{q}{M_0}\sin(\delta t+\delta_0)\Bigg[kc_1\sin(\omega_k t)\\&+kc_2\cos(\omega_k t)+kr_k\sin(\delta t+\delta_0)\\&+\int dk' a_{k'}^{(1)}\frac{(k^2-k'^2)}{4\omega_k\omega_{k'}\sinh(\pi\Delta/2)}\Bigg]\\&-\frac{q}{M_0\delta}[\cos(\delta t+\delta_0)-\cos(\delta_0)]\\&\times\left(k\dot{a}_k^{(1)}+\int dk'\dot{a}_{k'}^{(1)}\frac{(k^2-k'^2)}{4\omega_k\omega_{k'}\sinh(\pi\Delta/2)}\right)\\&+\frac{1}{2}\int dk_1\int dk_2 a_{k_1}^{(1)}a_{k_2}^{(1)}I_3(k_1,k_2,k).\end{aligned}\tag{12}$$

It can be seen from Eq. (12) that when we drive the original system with an ac force of frequency δ, the odd phonons ''feel'' an external force of frequency 2δ, so the resonances arise when $\delta\approx\omega_k/2$ [8]. Analogously with $a_k^{(1)}$, if we take into account the FBC, we obtain that $b_k^{(2)}$ also satisfies

$$b_k^{(2)}\{\cos(kL/2)[k^2+\cosh^{-2}(L/2)]-k\sin(kL/2)\tanh(L/2)\}=0,\tag{13}$$

whose solution is given by

$$k_m=\frac{2(m-1)\pi}{L},\tag{14}$$

for small integer $m(m\ll L)$ and $b_k^{(2)}=0$ otherwise. So, the resonances at $\delta\approx\omega_k/2$ are related with the excitation of the odd phonon modes. Combining Eqs. (11) and (14) we find that the allowed wave number for the partial differential equation (PDE) (1) with FBC are

$$k_n=\frac{(n-1)\pi}{L},\quad n=1,2,\ldots N,\quad n\ll L,\tag{15}$$

where the odd (even) values of n are related with the odd (even) phonon modes. Moreover, these results are consistent with the numerical findings of Ref. [1].

III. INTERPRETATION OF THE RESULTS

In analyzing the results of the perturbation theory, we observe that to first (linear) order, the effect of the ac driving in exciting the phonon modes is independent of the nonlinear wave (the kink) but rather related to the ''integrated strength'' of the mode that is to be excited [notice the term N_2 in Eq. (6)]. Consequently, as odd modes bear vanishing integrated strength, for $\delta\approx\omega_k$, only resonances with the even phonon modes can be identified (to leading order). On the contrary, higher order resonances involve the nonlinear wave [notice the term N_1 in Eq. (6)]. In the latter case, as we observe in Figs. 2 and 3, the excitation of modes that respect the symmetry of the nonlinear wave becomes preferrential (as we would expect on symmetry grounds). Hence for $\delta\approx\omega_k/2$, the excitation of phonon modes that have the same parity as the wave will be favored. In Fig. 1 we show the first odd and the first even phonon mode eigenfunction profiles, while in Figs. 2 and 3 we show their respective effects on the motion of the kink. Notice that the former results in a breathing type oscillation of the whole kink profile, while the latter affects only the steady states on the background of which the

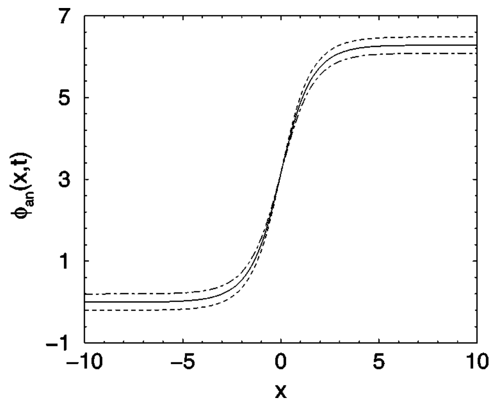

FIG. 2. Effect of the odd phonon modes on the sG kink: with a solid line we represent the static kink, ϕ_0, which is an exact solution of the sG equation. Under the effect of the first odd phonon mode [linear superposition of the sG kink and the function $b_1G_1(x)$, with $b_1=\pm0.5$, see Eq. (2)], the sG kink oscillates between the two profiles represented by the dashed and dash-dotted lines.

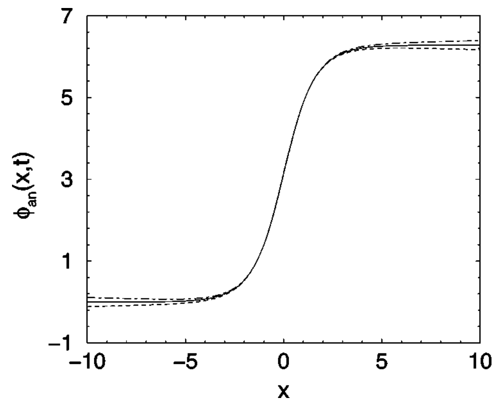

FIG. 3. Effect of the even phonon modes on the sG kink: with a solid line we represent the static kink, ϕ_0, which is an exact solution of the sG equation. Under the effect of the first even phonon mode [linear superposition of the sG kink and the function $a_2F_2(x)$, with $a_2=\pm1$, see Eq. (2)], the sG kink oscillates between the two profiles represented by the dashed and dash-dotted lines.

kink ''lives.'' The latter observation justifies the remarks above, as the first order excitation (at $\delta\approx\omega_k$) of the even modes is related only to the steady states (unrelated to the kink), while the second order excitation of the odd modes (at $\delta\approx\omega_k/2$) is predominantly related to the kink (rather than to the steady states). Notice that we can extend the obtained results by including the effect of damping in the system. It is natural to take into account the dissipative effects in the realistic cases of interest in applications. Furthermore it should be noted that in this way the divergences in the solutions of the Eqs. (7) and (12) are avoided and the perturbation theory is valid even when $\delta\approx\omega_k,\omega_k/2$.

IV. CONCLUSIONS

We have shown by using a linear perturbation theory that the ac force excites only the odd phonon modes at driving frequencies $\delta\approx\omega_k/2$. These modes of odd parity are energetically favored as they respect the symmetry of the kink. It was also shown that the even phonon modes (of even spatial parity) can be excited at driving frequencies $\delta\approx\omega_k$. So, at least in this problem ''not all phonons are equivalent.'' We should remark that this phenomenon is not only restricted to the action of the ac force in the sG system, but is, in fact, more general: Indeed, taking $\delta=0$ and $\delta_0=\pi/2$ in Eqs. (7) and (8), one can show that the nonequivalence of the phonon modes also arises when the system is driven by a dc field or even if we start from a distorted kink in the numerical simulations [9].

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